**Renormalization Group**

Okay let’s do a calculation. We’ll look at the renormalization of a 1D and 2D Ising Ferromagnet using the Kadanoff decimation approach. Then in the next file we’ll look at momentum shell renormalization, which we’ll find is a more accurate way to do things.

**RG analysis of the nearest neighbor Ising model in 1D**

Let’s start with the 1D Ising model,



whereby,



and Si is same as σi here.

**The way RG is to proceed in theory**

Assuming a lattice spacing of *a*, we group spins in a volume (ba)d together, where b is some number.



and the ζ factor there is to scale the spin back down to unit magnitude. And now we want to separate the partition function sum over states/spins into two parts. The first is a sum over Sik consistent with a certain value of k, and then a separate sum of all values of k. This is delineated below.



The second line comes from summing over all Si configurations, but restricting them so that the sum over the kth block is equal to k. So we can say that summing over all spins Si is the same as summing over all spins subject to the requirement that the sum over each block is equal to k, followed by a sum over all k. Now we don’t know what the new will look like. But we assume we’re ‘lucky’ and that the only change to the form of H will be to the coupling constants,



So now we have a new hamiltonian describing spins on a lattice spacing ã = ba. And the coupling constants will be renormalized to:



**But the way we’re really going to do RG**

So consider a nearest neighbor Ising model again,



and sum over every other spin, obtaining the b = 2 model. This procedure is called *decimation*. Let’s do a three spin example for simplicity.



where in the last line I neglect the ln2 constant as irrelevant to the Free Energy. Now we need to put this in the form of our original Hamiltonian. So use the fact that,



and expand for small j. Therefore,



Now we have to work out what those two terms are. Note we cannot just presume S1 and S3 are ‘small’, because they’re not really. So basically we want to Taylor expand these terms out to all orders, making note that tanh() involves only odd powers, and ln[cosh()] only even powers.



and keep just the S1, S3, and S1S3 terms, noting that S1,3even\_power = 1, S1,3odd\_power = S1,3. So to help with this, note that for n = even, we have:



Now the two sums in the { } have the same value. To see this, let S1 = 1, and S3 = -1. Since (1-1)n = 0, can see those two sums are the same. And then if we let S1 = S3 = 1, then can see that they must add up to 2n. So each individual sum is 2n/2 = 2n-1. So we have,



What if n is odd? Then,



By the same arguments each sum in the brackets must be equal to 2n-1. So we have:



So now we have:



and,



Filling this back into our Z,



discarding an irrelevant constant. Now have to be careful to generalize to N spins. Note that the S2 summation generated an effective interaction between S1,3, and an effective j-potential for S1,3 as well. An S4 summation would generate an effective interaction between S3,5 as well as an effective j-potential for S3,5. So when we do the summation over even spins, the odd spins will get two effective j-interaction terms. Accounting for this, we see that we’ll get:



1. Determine the fixed points,

So what are our scaling equations? Well after n decimations, we’d have:



In order to determine the critical point, we will want to determine the K, j which don’t change when scaled. So we look for solutions to:



The fixed point (and critical point in this case) solutions are:



K\* = ∞ (T = 0), gives the unstable fixed point and represents the critical point. So there is no finite-temperature critical point. This is to be expected in 1D. Further, we can see that the coupling constants flow to high T and high h, which definitely makes ξ smaller, as it must.

2. Expand equation around fixed points (to linear order in the variables only) and solve.

There are no critical exponents, but nonetheless, we can determine the scaling of the free energy, the magnetization, and the correlation length in the vicinity of the critical point; we would solve for K(b), and j(b). We’ll go back to:



and simplify around the fixed points K = ∞, and j = 0.



We can solve these equations,



and,



Further, the factor by which the effective lattice spacing grows is: b = 2n-1 → n = 1 + ln(b/2). So putting in terms of b,



Scaling looks like this:

Diagram

Description automatically generated

I guess we could say that as we scale out, the fluctuations would seem less severe, and so the effective temperature should be higher (which, as we can see when calculate GF, makes fluctuation responses lower) and the field should be stronger, as a stronger field would also mitigate fluctuations, as is easy to see in the infinite field limit.

3. Extract exponents from scaling relationships.

There are no critical exponents again. But we’ll look at how f, ξ scales nonetheless. Consider ξ. We said that it was given by (K = βJ, j = βh),



(Say we specialize to j = 0. Then if we set K = (0.5)ln(b/2), we’ll have:



So,



which is what we had obtained by other arguments. Note how the correlation length goes to 1 as the temperature is increased. This is to be expected since this would indicate that only spins within 1 unit cell (but there is only 1 spin in each unit cell…) are correlated. So basically, no spins are correlated. And of course it goes to ∞ as T → 0. Let’s look at the free energy density (×β). In the RG file, we were looking at the scaling of f = βF → βcF near the critical point. But now we have no non-zero critical point so β doesn’t go to βc, but rather to ∞. So we have to consider βF together. I’ll write this as (βF).



and let K – (1/2)ln(b/2) = 0 again. Then we have:



Well, we’re in 1D, so we should have:



And let’s compare to what we found when we exactly solved the Ising model previously in this folder. There we had:



So the scaling approximately holds. Just missing that extra -K = -J/T term. But it is in fact unimportant to the thermodynamic properties, since once we divide each side by β, to get F, the extra term becomes just a constant. And now let’s look at the magnetization. This would be:



Again, we’re in 1D, so the magnetization would be:



**RG analysis of the nearest neighbor Ising model in 2D**

So we see that there is no transition in 1D, as had been previously argued. Let’s consider the 2D case. Now it’s problematic to sum over ‘every other spin’ in this case. So what they do is replace the 2D model on the left with the one on the right.



The one on the left has each atom bonded to its 4 n.n. and so there is a total of 4J bonds on each atom. But the one on the right has only 2 n.n. per atom. So in order to keep the net bond strength at 4J, we double the right model’s individual bond strength to 2J. Most importantly, in the new one, you can verify that summing over the four n.n. of a spin at an intersection will deliver the same model but with double the lattice spacing. So it will easily renormalize. Further, we can see (can I?) that we can view it simply as our last 1D model, but ‘squared’. So we can generalize our previous results to quickly do the decimation over spins, and obtain the new coupling constants. We find:



1. Determine the fixed points,

Now we should get fixed points since there is a change in phase in 2D. So what are our scaling equations? Well after n decimations, we’d have:



In order to determine the critical point, we will want to determine the K, j which don’t change when scaled. So we look for solutions to:



The solutions are:



So this time we have non-zero fixed point (which will be our critical point)!

2. Expand equation around fixed points (to linear order only) and solve.

Expanding to linear order about the fixed point…



so…



and now solving…



Setting b = 2n-1,



which is the same as,



and so,



Flows look like:

Diagram

Description automatically generated

So when we start at a temperature below the critical point (I guess that’d mean K > K\* = Kc), and scale by factor b, then the effective temperature will decrease. And this makes sense because fluctuations are largest near critical point, and we’ll appear less severe as scale out, and so temperature should move away from critical point.

3. Extract exponents from scaling relationships

And so now we have,



And so all critical exponents follow from above. We see that the estimates for the new critical exponents are improvements upon MFT.



And all the other guys follow from:



In 2D, we actually have (from Onsager),

Table

Description automatically generated

So at the moment, except for ν, agreement got way worse actually. Hmmmm. Well. Moving on to the other way to do things.